Automorphisms of the Weyl algebra

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Abstract. We discuss a conjecture which says that the automorphism group of the Weyl algebra in characteristic zero is canonically isomorphic to the automorphism group of the corresponding Poisson algebra of classical polynomial symbols. Several arguments in favor of this conjecture are presented, all based on the consideration of the reduction of the Weyl algebra to positive characteristic.

1 Introduction

This paper is devoted to the following surprising conjecture.

Conjecture 1 *The automorphism group of the Weyl algebra of index n over $\mathbb{C}$ is isomorphic to the group of the polynomial symplectomorphisms of a $2n$-dimensional affine space*

$$\mathrm{Aut}(A_{n,\mathbb{C}}) \simeq \mathrm{Aut}(P_{n,\mathbb{C}}) \ .$$

Here for an integer $n \geq 1$, denote by $A_{n,\mathbb{C}}$ the Weyl algebra of index n over $\mathbb{C}$

$$\mathbb{C}\langle \hat{x}_1, \ldots, \hat{x}_{2n} \rangle / (\text{ relations } [\hat{x}_i, \hat{x}_j] = \omega_{ij}\ , \ \ 1 \leq i, j \leq 2n\ )$$

where $(\omega_{ij})_{1 \leq i,j \leq 2n}$ is the standard skew-symmetric matrix:

$$\omega_{ij} = \delta_{i,n+j} - \delta_{n+i,j}\ ,$$

and by $P_{n,\mathbb{C}}$ the Poisson algebra over $\mathbb{C}$ which is the usual polynomial algebra $\mathbb{C}[x_1, \ldots, x_{2n}] \simeq \mathcal{O}(\mathbb{A}^{2n}_{\mathbb{C}})$ endowed with the Poisson bracket:

$$\{x_i, x_j\} = \omega_{ij}\ ,\ 1 \leq i, j \leq 2n\ .$$

The algebra $A_{n,\mathbb{C}}$ is isomorphic to the algebra $\mathcal{D}(\mathbb{A}^n_{\mathbb{C}})$ of polynomial differential operators in n variables $x_1, \ldots, x_n$:

$$\hat{x}_i \mapsto x_i,\ \ \hat{x}_{n+i} \mapsto \partial_i := \partial/\partial x_i,\ \ 1 \leq i \leq n\ .$$

The conjecture becomes even more surprising if one takes into account the fact that the *Lie algebras* of derivations of $A_{n,\mathbb{C}}$ and $P_{n,\mathbb{C}}$ are not isomorphic to each other (see Section 3). Conjecture 1 is closely related to a question raised by one of us few years ago (see section 4.1, Question 4 in [4]), the motivation at that time came from the theory of deformation quantization for algebraic varieties.

One of main results of the present paper is Theorem 1 which says that the subgroups of $\mathrm{Aut}(A_{n,\mathbb{C}})$ and $\mathrm{Aut}(P_{n,\mathbb{C}})$ consisting of the so-called *tame* automorphisms, are naturally isomorphic to each other. Another result is Theorem 2 from Section 6.3, which allows to propose a (hypothetical) specific candidate for the isomorphism between two automorphism groups as above. The key idea is to use reduction to finite characteristic and the fact that the Weyl algebra in finite characteristic has a huge center isomorphic to the polynomial algebra. Another application of this idea is a proof of a stable equivalence between the Jacobian and Dixmier conjectures, see [1]. This paper is an extended version of the talk given by one of us (see [5]) on Arbeitstagung 2005 (Bonn).

We finish the introduction with

1.1 First positive evidence: case $n = 1$

The structure of the group $\mathrm{Aut}(P_{1,\mathbb{C}})$ is known since the work of H. W. E. Jung (see [3]). This group contains the group $G_1 = SL(2, \mathbb{C}) \ltimes \mathbb{C}^2$ of special affine transformations, and the solvable group G_2 of polynomial transformations of the form

$$(x_1, x_2) \mapsto (\lambda x_1 + F(x_2), \lambda^{-1} x_2),\ \ \lambda \in \mathbb{C}^\times,\ \ F \in \mathbb{C}[x]\ .$$

The group $\mathrm{Aut}(P_{1,\mathbb{C}})$ is equal to the amalgamated product of G_1 and G_2 over their intersection. J. Dixmier in [2] and later L. Makar-Limanov in [6]

proved that if one replaces the commuting variables x_1, x_2 by noncommuting variables $\hat{x}_1, \hat{x}_2$ in the formulas above, one obtains the description of the group $\mathrm{Aut}(A_{1,\mathbb{C}})$. Hence, in the case $n = 1$ the two automorphism groups are isomorphic.

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2 Automorphism groups as ind-schemes

For an arbitrary commutative ring R one can define the Weyl algebra $A_{n,R}$ over R, by just replacing $\mathbb{C}$ by R in the definition. We denote the algebra $A_{n,\mathbb{Z}}$ simply by A_n, hence $A_{n,R} = A_n \otimes R$. The algebra $A_{n,R}$ considered as an R-module is free with basis

$$\hat{x}^\alpha := \hat{x}_1^{\alpha_1} \dots \hat{x}_{2n}^{\alpha_{2n}},\ \alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{Z}_{\geq 0}^{2n} \ .$$

We define an increasing filtration (the Bernstein filtration) on the algebra $A_{n,R}$ by

$$A_{n,R}^{\leq N} := \left\{ \sum_\alpha c_\alpha \hat{x}^\alpha \,|\, c_\alpha \in R,\ c_\alpha = 0 \text{ for } |\alpha| := \alpha_1 + \dots + \alpha_{2n} > N \right\} \ .$$

This filtration induces a filtration on the automorphism group:

$$\mathrm{Aut}^{\leq N} A_{n,R} := \{ f \in \mathrm{Aut}(A_{n,R}) |\, f(\hat{x}_i), f^{-1}(\hat{x}_i) \in A_{n,R}^{\leq N} \ \forall i = 1, \dots, 2n \} \ .$$

The following functor on commutative rings:

$$R \mapsto \mathrm{Aut}^{\leq N}(A_{n,R}) \ ,$$

is representable by an affine scheme of finite type over $\mathbb{Z}$. We denote this scheme by

$$\underline{\mathrm{Aut}}^{\leq N}(A_n) \ .$$

The ring of functions $\mathcal{O}(\underline{\mathrm{Aut}}^{\leq N}(A_n))$ is generated by the variables

$$(c_{i,\alpha}, c'_{i,\alpha})_{1 \leq i \leq 2n, |\alpha| \leq N}$$

which are the coefficients of the elements $f(\hat{x}_i), f^{-1}(\hat{x}_i)$ in the standard basis $(\hat{x}^\alpha)$ of the Weyl algebra.

The obvious inclusions $\underline{\mathrm{Aut}}^{\leq N}(A_n) \hookrightarrow \underline{\mathrm{Aut}}^{\leq (N+1)}(A_n)$ are closed embeddings, the inductive limit over N of schemes $\underline{\mathrm{Aut}}^{\leq N}(A_n)$ is an ind-affine scheme over $\mathbb{Z}$. We denote it by $\underline{\mathrm{Aut}}(A_n)$, it is a group-like object in the category of ind-affine schemes.

Similarly, one can define all the above notions for the Poisson algebra P_n, in particular we have an affine scheme $\underline{\mathrm{Aut}}^{\leq N}(P_n)$ of finite type, and a group ind-affine scheme $\underline{\mathrm{Aut}}(P_n)$.

Later it will be convenient to use the notation

$$\underline{\mathrm{Aut}}^{\leq N}(A_{n,R}) := \underline{\mathrm{Aut}}^{\leq N}(A_n) \times_{\mathsf{Spec}\,\mathbb{Z}} \mathsf{Spec}\, R$$

(here R is an arbitrary commutative ring), for a scheme over $\mathsf{Spec}\, R$ obtained by the extension of scalars, similarly we will have schemes $\underline{\mathrm{Aut}}^{\leq N}(P_{n,R})$ for the case of Poisson algebras.

There is also another sequence of closed embeddings (stabilization)

$$\underline{\mathrm{Aut}}^{\leq N}(A_n) \hookrightarrow \underline{\mathrm{Aut}}^{\leq N}(A_{n+1}),\ \underline{\mathrm{Aut}}^{\leq N}(P_n) \hookrightarrow \underline{\mathrm{Aut}}^{\leq N}(P_{n+1})$$

corresponding to the addition of two new generators and extending the automorphisms by the trivial automorphism on the additional generators.

Conjecture 1 says that groups of points $\underline{\mathrm{Aut}}(A_n)(\mathbb{C})$ and $\underline{\mathrm{Aut}}(P_n)(\mathbb{C})$ are isomorphic. We expect that the isomorphism should preserve the filtration by degrees, compatible with stabilization embeddings, and should be a constructible map for any given term of filtration, defined over $\mathbb{Q}$:

Conjecture 2 *There exists a family $\phi_{n,N}$ of constructible one-to-one maps*

$$\phi_{n,N} : \underline{\mathrm{Aut}}^{\leq N}(A_{n,\mathbb{Q}}) \to \underline{\mathrm{Aut}}^{\leq N}(P_{n,\mathbb{Q}})$$

compatible with the inclusions increasing indices N and n, and with the group structure.

Obviously, this conjecture implies Conjecture 1, and moreover it implies that the isomorphism exists if one replaces $\mathbb{C}$ by an arbitrary field of characteristic zero.

3 Negative evidence

For any group object G in the category of ind-affine schemes over $\mathbb{Q}$, one can associate its Lie algebra $\mathrm{Lie}(G)$, by considering points of G with coefficients in the algebra of dual numbers $\mathbb{Q}[t]/(t^2)$. The Lie algebras

$$\mathrm{Lie}(\underline{\mathrm{Aut}}(A_{n,\mathbb{Q}})), \ \ \mathrm{Lie}(\underline{\mathrm{Aut}}(P_{n,\mathbb{Q}}))$$

are by definition the algebras of derivations of $A_{n,\mathbb{Q}}$ and $P_{n,\mathbb{Q}}$, respectively. The following fact is well known:

Lemma 1 *All derivations of the Weyl algebra are inner:*

$$\mathrm{Der}(A_{n,\mathbb{Q}}) \simeq A_{n,\mathbb{Q}}/\mathbb{Q}\cdot 1_{A_{n,\mathbb{Q}}}, \ \ f \in A_{n,\mathbb{Q}} \mapsto [f,\cdot] \in \mathrm{Der}(A_{n,\mathbb{Q}}) \ .$$

Proof: For any algebra A the space of its outer derivations coincides with the first Hochschild cohomology,

$$H^1(A,A) = \mathrm{Ext}^1_{A-mod-A}(A,A) \ .$$

We claim that for $A = A_{n,\mathbb{Q}}$ the whole Hochschild cohomology is a 1-dimensional space in degree zero. Namely, it is easy to see that there exists an isomorphism $A \otimes A^{op} \simeq \mathcal{D}(\mathbb{A}^{2n}_{\mathbb{Q}})$ such that the diagonal bimodule A is isomorphic to $\mathcal{O}(\mathbb{A}^{2n}_{\mathbb{Q}})$. The result follows from a standard property of $\mathcal{D}$-modules:

$$\mathrm{Ext}^*_{\mathcal{D}_X-mod}(\mathcal{O}_X,\mathcal{O}_X) = H^*_{de\ Rham}(X)$$

which holds for any smooth variety X. $\square$

Derivations of $P_{n,\mathbb{Q}}$ are polynomial Hamiltonian vector fields:

$$\mathrm{Der}(P_{n,\mathbb{Q}}) \simeq P_{n,\mathbb{Q}}/\mathbb{Q}\cdot 1_{P_{n,\mathbb{Q}}}, \ \ f \in P_{n,\mathbb{Q}} \mapsto \{f,\cdot\} \in \mathrm{Der}(P_{n,\mathbb{Q}}) \ .$$

We see that both Lie algebras of derivations are of the "same" size, each of them has a basis labeled by the set $\mathbb{Z}^{2n}_{\geq 0} \setminus \{(0,\dots,0)\}$. Nevertheless, these two Lie algebras are not isomorphic. Namely, $\mathrm{Der}(P_{n,\mathbb{Q}})$ has many nontrivial Lie subalgebras of finite codimension (e.g. Hamiltonian vector fields vanishing at a given point in $\mathbb{Q}^{2n}$), whereas the algebra $\mathrm{Der}(A_{n,\mathbb{Q}})$ has no such subalgebras.

The conclusion is that the hypothetical constructible isomorphism $\phi_{n,N}$ cannot be a scheme map.

4 Positive evidence: tame automorphisms

Obviously, the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ acts by automorphisms of $A_{n,\mathbb{C}}$ and $P_{n,\mathbb{C}}$, by symplectic linear transformations of the variables $\hat{x}_1,\dots,\hat{x}_{2n}$

and $x_1, \ldots, x_{2n}$, respectively. Also, for any polynomial $F \in \mathbb{C}[x_1, \ldots, x_n]$ we define "transvections"

$$T_F^A \in \mathrm{Aut}(A_{n,\mathbb{C}}),\ T_F^P \in \mathrm{Aut}(P_{n,\mathbb{C}})$$

by the formulas

$$T_F^P(x_i) = x_i,\ T_F^P(x_{n+i}) = x_{n+i} + \partial_i F(x_1, \ldots, x_n),\ 1 \leq i \leq n\ ,$$

$$T_F^A(\hat{x}_i) = \hat{x}_i,\ T_F^A(\hat{x}_{n+i}) = \hat{x}_{n+i} + \partial_i F(\hat{x}_1, \ldots, \hat{x}_n),\ 1 \leq i \leq n\ .$$

The last formula makes sense, as we substitute the commuting variables $\hat{x}_1, \ldots, \hat{x}_n$ in place of $x_1, \ldots, x_n$ in the polynomial $\partial_i F := \partial F / \partial x_i$. A straightforward check shows that these maps are well defined, i.e. T_F^P preserves the Poisson bracket and T_F^A preserves the commutation relations between the generators. The automorphism T_F^A is in a sense the conjugation by a nonalgebraic element $\exp(F(\hat{x}_1, \ldots, \hat{x}_n))$.

The correspondence $F \mapsto T_F^A$ (resp. $F \mapsto T_F^P$) gives a group homomorphism $\mathbb{C}[x_1, \ldots, x_n]/\mathbb{C} \cdot 1 \to \mathrm{Aut}(A_{n,\mathbb{C}})$ (resp. to $\mathrm{Aut}(P_{n,\mathbb{C}})$).

Let us denote by G_n the free product of $\mathrm{Sp}(2n, \mathbb{C})$ with the abelian group $\mathbb{C}[x_1, \ldots, x_n]/\mathbb{C} \cdot 1$. We obtain two homomorphisms ρ_n^A and ρ_n^P from G_n to $\mathrm{Aut}(A_{n,\mathbb{C}})$ and $\mathrm{Aut}(P_{n,\mathbb{C}})$ respectively. The automorphisms which belong to the image of ρ_n^A (resp. ρ_n^P) are called *tame*.

Theorem 1 *In the above notation,* $\mathrm{Ker}\,\rho_n^A = \mathrm{Ker}\,\rho_n^P$.

As an immediate corollary, we obtain that the groups of tame automorphisms of $A_{n,\mathbb{C}}$ and $P_{n,\mathbb{C}}$ are canonically isomorphic. It is quite possible that all of the automorphisms of $A_{n,\mathbb{C}}$ and $P_{n,\mathbb{C}}$ become tame after stabilization, i.e. after adding several dummy variables and increasing the parameter n. If this is the case, then we obtain Conjecture 1 (and, in fact, Conjecture 2 as well).

Remark 1 *We expect that the canonical isomorphism in Conjecture 2 coincides with the above isomorphism on subgroups of tame elements.*

The proof of Theorem 1 will be given in Section 7.

Remark 2 *I.R.Shafarevich in [9] introduced a notion of a Lie algebra for an infinite-dimensional algebraic group (in fact, his notion of a group is a*

bit obscure as he does not use the language of ind-schemes). It is known that the Lie algebra associated with the group of all of the automorphisms of a polynomial ring coincides with the Lie algebra of the group of tame *automorphisms. The same is true for the automorphisms of the Weyl algebra and polynomial symplectomorphisms. However, I.P.Shestakov and U.Umirbaev [10] have shown that the Nagata automorphism is wild. Hence, we have an infinite-dimensional effect: A proper subgroup can have the same Lie algebra as the whole group. Our Conjecture indicates that further pathologies are possible, the same group has two different Lie algebras when interpreted as an ind-scheme in two different ways. Presumably, it means that at least one of our automorphism groups is singular everywhere.*

5 The Weyl algebra in finite characteristic

Here we introduce the main tool which allows us to relate the algebras $A_{n,R}$ and $P_{n,R}$ (and their automorphisms).

5.1 The Weyl algebra as an Azumaya algebra

Let R be a commutative ring in characteristic $p > 0$, i.e. $p \cdot 1_R = 0 \in R$. It is well known that in this case the Weyl algebra $A_{n,R}$ has a big center, and moreover, it is an Azumaya algebra of rank p^n (see [8]).

Proposition 1 *For any commutative ring $R \supset \mathbb{Z}/p\mathbb{Z}$ the center $C_{n,R}$ of $A_{n,R}$ is isomorphic as an R-algebra to the polynomial algebra $R[y_1, \dots, y_{2n}]$, where the variable $y_i,\ i = 1, \dots, 2n$, corresponds to $\hat{x}_i^p$. The algebra $A_{n,R}$ is a free $C_{n,R}$-module of rank p^{2n}, and it is an Azumaya algebra of rank p^n over $C_{n,R}$.*

Proof: First of all, a straightforward check shows that the elements $(\hat{x}_i^p)_{1 \le i \le 2n}$ are central and generate over R the polynomial algebra. The algebra $A_{n,R}$ is the algebra over $R[y_1, \dots, y_{2n}]$ with generators $\hat{x}_1, \dots, \hat{x}_{2n}$ and relations

$$[\hat{x}_i, \hat{x}_j] = \omega_{ij},\ \hat{x}_i^p = y_i,\ 1 \le i, j \le 2n \ .$$

After the extension of scalars from $R[y_1, \dots, y_{2n}]$ to $C'_{n,R} := R[y_1^{1/p}, \dots, y_{2n}^{1/p}]$ the pullback of the Weyl algebra can be described as an algebra over $C'_{n,R}$

with generators $\hat{x}'_i := \hat{x}_i - y_i^{1/p}$ and relations

$$[\hat{x}'_i, \hat{x}'_j] = \omega_{ij},\ (\hat{x}'_i)^p = 0,\ 1 \le i, j \le 2n\ .$$

It is well known that the algebra over $\mathbb{Z}/p\mathbb{Z}$ with two generators $\hat{x}_1, \hat{x}_2$ and defining relations $[\hat{x}_1, \hat{x}_2] = 1$, $\hat{x}_1^p = \hat{x}_2^p = 0$ is isomorphic to the matrix algebra $\mathrm{Mat}(p \times p, \mathbb{Z}/p\mathbb{Z})$ (consider operators x and d/dx in the truncated polynomial ring $\mathbb{Z}/p\mathbb{Z}[x]/(x^p)$). Hence, after the faithfully flat finitely generated extension from $R[y_1, \dots, y_{2n}]$ to $C'_{n,R}$, we obtain the matrix algebra $\mathrm{Mat}(p^n \times p^n, C'_{n,R})$. Then the proposition follows from standard properties of Azumaya algebras. $\square$

5.2 Poisson bracket on the center of the Weyl algebra

The next observation is that for any commutative ring R flat over any prime $p \in \mathsf{Spec}\,\mathbb{Z}$ one can define a Poisson bracket on $C_{n,R/pR}$ in an intrinsic manner. Namely, for such R and any two elements $a, b \in C_{n,R/pR}$ we define the element $\{a, b\} \in C_{n,R/pR}$ by the formula

$$\{a, b\} := \frac{[\tilde{a}, \tilde{b}]}{p} \pmod{pR}\ ,$$

where $\tilde{a}, \tilde{b} \in A_{n,R}$ are arbitrary lifts of

$$a, b \in C_{n,R/pR} \subset A_{n,R/pR} = A_{n,R} \pmod{pR}\ .$$

A straightforward check shows that the operation $(a, b) \mapsto \{a, b\}$ is well defined, takes values in $C_{n,R/pR} \subset A_{n,R/pR}$, satisfies the Leibniz rule with respect to the product on $C_{n,R/pR}$, and the Jacobi identity.

Morally, our construction of the bracket is analogous to the well-known counterpart in deformation quantization. If one has a one-parameter family of associative algebras $A_\hbar$ (flat over the algebra of functions of $\hbar$), then on the center of A_0 one has a canonical Poisson bracket given by the "same" formula as above:

$$\{a_0, b_0\} := \frac{[a_\hbar, b_\hbar]}{\hbar} + O(\hbar)\ ,$$

where $a_\hbar, b_\hbar \in A_\hbar$ are arbitrary extensions of elements

$$a_0, b_0 \in \mathrm{Center}(A_0) \subset A_0\ .$$

The prime number p plays the role of Planck constant $\hbar$.

The following lemma shows that the canonical Poisson bracket on $C_{n,R/pR}$ coincides (up to sign) with the standard one:

Lemma 2 *In the above notation, one has*

$$\{y_i, y_j\} = -\omega_{ij} \ .$$

Proof: It is enough to make the calculation in the case of one variable. The following elementary identity holds in the algebra of differential operators with coefficients in $\mathbb{Z}$:

$$[(d/dx)^p, x^p] = \sum_{i=0}^{p-1} \frac{(p!)^2}{(i!)^2(p-i)!} x^i (d/dx)^i \ .$$

The r.h.s. is divisible by p, and is equal to $-p$ modulo p^2. $\square$

In the above considerations one can make a weaker assumption, it is enough to consider the coefficient ring R flat over $\mathbb{Z}/p^2\mathbb{Z}$. The corollary is that for any automorphism of the Weyl algebra in characteristic p which admits a lift $\mod p^2$, the induced automorphism of the center preserves the canonical Poisson bracket. The condition of the existence of the lift is necessary. For example, the automorphism of $A_{2,\mathbb{Z}/p\mathbb{Z}}$ given by

$$\hat{x}_1 \mapsto \hat{x}_1 + \hat{x}_2^p \hat{x}_3^{p-1}, \quad \hat{x}_i \mapsto \hat{x}_i, \ i = 2, 3, 4$$

acts on the center by

$$y_1 \mapsto y_1 + y_2^p y_3^{p-1} - y_2, \quad y_i \mapsto y_i, \ i = 2, 3, 4 \ .$$

The above map does not preserve the Poisson bracket.

6 Correspondence between automorphisms in finite characteristic

6.1 Rings at infinite prime

It will be convenient to introduce the following notation ("reduction modulo infinite prime") for an arbitrary commutative ring R:

$$R_\infty := \varinjlim_{f.g.\, R' \subset R} \left(\prod_{\text{primes } p} R' \otimes \mathbb{Z}/p\mathbb{Z} \ \Big/ \ \bigoplus_{\text{primes } p} R' \otimes \mathbb{Z}/p\mathbb{Z} \right) \ .$$

Here the inductive limit is taken over the filtered system consisting of all finitely generated subrings of R, the index p runs over primes $2, 3, 5, \dots$.

It is easy to see that the ring R_∞ is defined over $\mathbb{Q}$ (all primes are invertible in R_∞), and the obvious map $R \mapsto R_\infty$ gives an inclusion $i : R \otimes \mathbb{Q} \hookrightarrow R_\infty$. Also, there is a universal Frobenius endomorphism $\mathrm{Fr} : R_\infty \to R_\infty$ given by

$$\mathrm{Fr}(a_p)_{\text{primes } p} := (a_p^p)_{\text{primes } p} \ .$$

Finally, if R has no nilpotents then $\mathrm{Fr} \circ i$ gives another inclusion of $R \otimes \mathbb{Q}$ into R_∞.

Remark 3 *Maybe a better notation would be $R \pmod{\infty}$ instead of R_∞ as one can also imitate p-adic completion:*

$$\varinjlim_{f.g.\, R' \subset R} \varprojlim_{n \geq 1} \left(\prod_{\text{primes } p} R' \otimes \mathbb{Z}/p^n\mathbb{Z} \ \Big/ \ \bigoplus_{\text{primes } p} R' \otimes \mathbb{Z}/p^n\mathbb{Z} \right) \ .$$

This larger ring has a canonical element "infinite prime" P which is the class of sequence $a_p = p \ \forall$ prime p. Our "reduction modulo infinite prime" is literally the reduction of the larger ring modulo P.

6.2 The homomorphism ψ_R

It follows from the previous section that one has a canonical group homomorphism

$$\psi_R : \underline{\mathrm{Aut}}(A_n)(R) \to \underline{\mathrm{Aut}}(P_n)(R_\infty) \ .$$

Namely, if $f \in \underline{\mathrm{Aut}}(A_n)(R)$ is an automorphism of $A_{n,R}$ then it belongs to a certain term of filtration $\underline{\mathrm{Aut}}^{\leq N}(A_n)(R)$ and moreover, is defined over a finitely generated ring $R' \subset R$. For any prime p, the automorphism f gives an automorphism f_p of $A_{n,R'/pR'}$, and hence an automorphism f_p^{centr} of the center $C_{n,R'/pR'} \simeq R'/pR'[y_1, \dots, y_{2n}]$.

Lemma 3 *For any $f \in \mathrm{Aut}^{\leq N}(A_n)(R)$ and any $i = 1, \dots, 2n$, the element $f_p^{centr}(y_i) \in C_{n,R'/pR'}$ is a polynomial in $y_1, \dots, y_{2n}$ of degree $\leq N$.*

Proof: One has $f_p^{centr}(y_i) = f_p(\hat{x}_i^p) = (f_p(\hat{x}_i))^p$. The element $f_p(\hat{x}_i) \in A_{n,R'/pR'}$ has degree $\leq N$ by our assumption. Hence, $f_p(\hat{x}_i^p)$ has degree $\leq pN$. We know that the last element is in fact a polynomial in the commuting variables $\hat{x}_1^p, \dots, \hat{x}_{2n}^p$. Then it is a polynomial of degree $\leq N$ in these variables,

as it follows immediately from the fact that $(\hat{x}^\alpha)_{\alpha\in\mathbb{Z}^{2n}_{\geq 0}}$ is a R'/pR' basis of $A_{n,R'/pR'}$. $\square$

Any finitely generated commutative ring is flat over all sufficiently large primes. Hence, we obtain polynomial symplectomorphisms f_p^{centr} for $p \gg 1$ of degree (and the degree of the inverse automorphism) uniformly bounded by N from above. We define $\psi_R(f)$ to be the collection $(f_p^{centr})_{p\gg 1}$ of automorphisms of $P_{n,R'/pR'}$ where we identify the variables $y_i \in C_{n,R'/pR'}$ with $x_i \in P_{n,R'/pR'}$, considered asymptotically in p. It is easy to see that $\psi_R(f)$ is an element of $\underline{\mathrm{Aut}}^{\leq N}(P_n)(R_\infty)$ and it does not depend on the choice of a finitely generated ring $R' \subset R$ over which f is defined. Hence we obtain a canonical map ψ_R which is obviously a group homomorphism.

6.3 Untwisting the Frobenius endomorphism

Theorem 2 *Let R be a finitely generated ring such that $\mathsf{Spec}\, R$ is smooth over $\mathsf{Spec}\,\mathbb{Z}$. Then for any $f \in \mathrm{Aut}(A_{n,R})$ the corresponding symplectomorphism $f_p^{centr} \in \mathrm{Aut}(P_{n,R/pR})$ is defined over $(R/pR)^p$ for sufficiently large p.*

Proof: First of all, notice that the subring $(R/pR)^p \subset R/pR$ coincides with the set of elements annihilated by all derivations of R/pR over $\mathbb{Z}/p\mathbb{Z}$ because R/pR is smooth over $\mathbb{Z}/p\mathbb{Z}$. Let us choose a finite collection

$$\delta_i \in \mathrm{Der}(R),\ i \in I,\ |I| < \infty$$

of derivations of R over $\mathbb{Z}$ such that for all sufficiently large p, the elements $\delta_i \pmod{p}$ span the tangent bundle $T_{\mathsf{Spec}\,(R/pR)/\mathsf{Spec}\,(\mathbb{Z}/p\mathbb{Z})}$. We have to prove that

$$\delta_i(f_p^{centr}(y_j)) = 0 \in C_{n,R/pR} \subset A_{n,R/pR}$$

for all $i \in I, j \in \{1,\ldots,2n\}$ and almost all prime numbers p. Let us (for given i,j,p) introduce the following notation:

$$a := f_p(\hat{x}_j),\ b := \delta_i(f_p(\hat{x}_j)) = \delta_i(a)\ .$$

Applying the Leibniz rule to the last expression in the next line

$$\delta_i(f_p^{centr}(y_j)) = \delta_i(f_p(\hat{x}_j^p)) = \delta_i((f_p(\hat{x}_j))^p) = \delta_i(a^p)\ ,$$

we conclude that we have to prove the equality

$$ba^{p-1} + aba^{p-2} + \cdots + a^{p-1}b = 0\ .$$

Notice that $\hat{x}_j$ is a locally ad-nilpotent element of $A_{n,R}$, i.e. for any $u \in A_{n,R}$ there exists $D = D(u) > 0$ such that

$$(\mathrm{ad}(\hat{x}_j))^k(u) = 0$$

for $k \geq D(u)$. Namely, one can take $D(u) = \deg(u) + 1$ where $\deg(u)$ is the degree of u in the Bernstein filtration. Using the assumption that f is an automorphism, we conclude that $f(\hat{x}_j)$ is again a locally ad-nilpotent element. In particular, there exists an integer $D \geq 0$ such that

$$(\mathrm{ad}(f(\hat{x}_j)))^D\,(\delta_i(f(\hat{x}_j))) = 0$$

for all i, j.

Finally, if the prime p is sufficiently large, $p - 1 \geq D$, then

$$0 = (\mathrm{ad}(a))^{p-1}(b) = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} a^i b\, a^{p-1-i} = \sum_{i=0}^{p-1} a^i b\, a^{p-1-i} \pmod{p}\ .$$

This finishes the proof. $\square$

The conclusion is that for a finitely generated algebra R smooth over $\mathbb{Z}$ there exists a unique homomorphism

$$\phi_R : \underline{\mathrm{Aut}}(A_n)(R) \to \underline{\mathrm{Aut}}(P_n)(R_\infty)$$

such that $\psi_R = \mathrm{Fr}_* \circ \phi_R$. Here $\mathrm{Fr}_* : \underline{\mathrm{Aut}}(P_n)(R_\infty) \to \underline{\mathrm{Aut}}(P_n)(R_\infty)$ is the group homomorphism induced by the endomorphism $\mathrm{Fr} : R_\infty \to R_\infty$ of the coefficient ring.

Conjecture 3 *In the above notation the image of ϕ_R belongs to*

$$\underline{\mathrm{Aut}}(P_n)(i(R) \otimes \mathbb{Q})\ ,$$

where $i : R \to R_\infty$ is the tautological inclusion (see Section 6.1). In other words, there exists a unique homomorphism

$$\phi_R^{can} : \underline{\mathrm{Aut}}(P_n)(R) \to \underline{\mathrm{Aut}}(P_n)(R \otimes \mathbb{Q})$$

such that $\psi_R = \mathrm{Fr}_* \circ i_* \circ \phi_R^{can}$.

If we assume that the above conjecture holds then we can define a constructible map

$$\phi^{can}_{n,N} : \underline{\mathrm{Aut}}^{\leq N}(A_{n,\mathbb{Q}}) \to \underline{\mathrm{Aut}}^{\leq N}(P_{n,\mathbb{Q}})$$

for arbitrary integers $n, N \geq 1$ in the following way. Let us decompose the scheme $\underline{\mathrm{Aut}}^{\leq N}(A_{n,\mathbb{Q}})$ into a finite union $\sqcup_{i\in I}\mathsf{Spec}\, R_i,\ |I| < \infty$ of closed affine schemes of finite type smooth over $\mathsf{Spec}\, Q$. Then choose a model $\mathsf{Spec}\, R'_i$ smooth over $\mathbb{Z}$ of each scheme $\mathsf{Spec}\, R_i$ such that R'_i is a finitely generated ring. The universal automorphism u_i of A_n defined over $\mathsf{Spec}\, R'_i$ maps under ϕ^{can}_R to a certain element v_i of $\underline{\mathrm{Aut}}^{\leq N}(P_n)(R_i)$. Taking the union of v_i, $i \in I$ we obtain a constructible map $\phi^{can}_{n,N}$. It is easy to see that this map does not depend on the choices made, and the correspondence $(n, N) \mapsto \phi^{can}_{n,N}$ is compatible with inclusions in the indices n, N. Moreover, the limiting map ϕ^{can}_n is compatible with the group structure. This is the (conjectured) *canonical* isomorphism between the two automorphism groups.

6.4 Continuous constructible maps

If we do not assume Conjecture 3, still the results of the previous section imply that for any $n, N \geq 1$ there exists $p_0(n, N) \geq 1$ such that for any prime $p > p_0(n, N)$ we have a canonically defined constructible map

$$\phi^{can}_{n,N,p} : \underline{\mathrm{Aut}}^{\leq N}(A_{n,\mathbb{Z}/p\mathbb{Z}}) \hookrightarrow \underline{\mathrm{Aut}}^{\leq N}(P_{n,\mathbb{Z}/p\mathbb{Z}})$$

defined by the property

$$\mathrm{Fr}_* \circ \phi^{can}_{n,N,p}(f) = f^{centr},\ \forall f \in \mathrm{Aut}^{\leq N}(A_n)(\mathbf{k})$$

for any field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = p$.

This map is an embedding because of the following lemma:

Lemma 4 *For any field* $\mathbf{k}$ *of characteristic* p *the map*

$$\mathrm{Aut}(A_{n,\mathbf{k}}) \to \mathrm{Aut}(\mathbb{A}^{2n}_{\mathbf{k}}),\ \ f \mapsto f^{centr}$$

is an inclusion.

Proof: The above map is a group homomorphism, hence it is enough to prove that any element $f \in \mathrm{Aut}(A_{n,\mathbf{k}})$ which is mapped to the identity map is the identity itself. Let us assume that $f^{centr} = \mathrm{Id}_{A^{2n}_{\mathbf{k}}}$ and $f \neq \mathrm{Id}_{A_{n,\mathbf{k}}}$. We consider two cases.

First case. There exists $N \geq 2$ such that $f(\hat{x}_i)$ has degree $\leq N$ for all i, and equal to N for some i. In this case the same will hold for f^{centr}, as the correspondence $f \mapsto f^{centr}$ preserves the filtration by degree (see Lemma 3) and is equal to the Frobenius map on the principal symbols with respect to the filtration. Hence we get a contradiction with the assumption $f^{centr} = \mathrm{Id}_{A^{2n}_{\mathbf{k}}}$.

Second case. The degree of $f(\hat{x}_i)$ is equal to 1 for all i, $1 \leq i \leq 2n$. In this case f is an affine symplectic map, and a direct calculation shows that the corresponding map f^{centr} is also affine symplectic with coefficients equal to the p-th power of those of f (it follows immediately from the results of the next section). Hence $f^{centr} = \mathrm{Id}_{A^{2n}_{\mathbf{k}}}$ implies $f = \mathrm{Id}_{A_{n,\mathbf{k}}}$, and we again get a contradiction. $\square$

Conjecture 4 *For any n, N there exists $p_1(n,N) \geq p_0(n,N)$ such that for any prime $p > p_1(n,N)$, the constructible map $\phi^{can}_{n,N,p}$ is a bijection.*

Obviously, Conjectures 3 and 4 together imply Conjecture 2.

It is easy to see that the map $\phi^{can}_{n,N,p}$ is *continuous* for the Zariski topology. It follows immediately from the fact that the correspondence $f \mapsto f^{centr}$ is a regular map (hence continuous) and that the Frobenius endomorphism of any scheme of finite type in characteristic $p > 0$ is a homeomorphism. It leads to a natural question whether the hypothetical canonical isomorphism $\phi^{can}_{n,N}$ is in fact a homeomorphism for the Zariski topology.

There exists a general notion of seminormalization ${}^+S$ for a reduced scheme S (see [12]). One of possible definitions (in the affine Noetherian case) is that a function f on ${}^+S$ is a reduced closed subscheme Z_f of $S \times \mathbb{A}^1$ which projects bijectively to S. Seminormalization is a tautological operation for smooth S, it coincides with the normalization for integral S. The above question about the homeomorphicity of $\phi^{can}_{n,N,p}$ (a strengthening of Conjecture 4) can be rephrased as follows:

Are seminormalizations of the reduced *schemes* $\left(\underline{\mathrm{Aut}}^{\leq N}(A_{n,\mathbb{Z}/p\mathbb{Z}})\right)^{red}$ *and* $\left(\underline{\mathrm{Aut}}^{\leq N}(P_{n,\mathbb{Z}/p\mathbb{Z}})\right)^{red}$ *isomorphic?*

7 Correspondence for tame automorphisms

Here we give a proof of Theorem 1. Moreover, we will show that the map ϕ^{can}_n is well defined on the tame automorphisms of $A_{n,\mathbb{C}}$, and it takes values in the group of tame automorphisms of $P_{n,\mathbb{C}}$.

First of all, we calculate the action of elementary tame automorphisms of the Weyl algebra in finite characteristic on its center.

Proposition 2 *Let* $\mathbf{k}$ *be a field of characteristic* $p > 2$ *and* $f \in \mathrm{Aut}(A_{n,\mathbf{k}})$ *be an automorphism given by a linear symplectic mapping on generators*

$$f(\hat{x}_i) = \sum_{j=1}^{2n} a_{ij}\hat{x}_j, \ \ a_{ij} \in \mathbf{k} \ .$$

Then the corresponding automorphism of the center $C_{n,\mathbf{k}} \simeq \mathbf{k}[y_1, \ldots, y_{2n}] = \mathbf{k}[\hat{x}_1^p, \ldots, \hat{x}_{2n}^p]$ *is given by*

$$f^{centr}(y_i) = \sum_{j=1}^{2n} (a_{ij})^p y_j \ .$$

Proof: The symplectic group $\mathrm{Sp}(2n, \mathbf{k})$ is generated by transvections

$$\hat{x}_1 \mapsto \hat{x}_1 + a\hat{x}_{n+1}, \ a \in \mathbf{k}, \ \ \xi_i \mapsto \xi_i \text{ for } i \geq 2$$

and by the Weyl group (Coxeter group C_n). The correspondence is obvious for elements of the Weyl group, and follows from the next Proposition for generalized transvections applied to a polynomial of degree 2. $\square$

Proposition 3 *Let* $\mathbf{k}$ *be a field of characteristic* p *and* $f = T_F^A \in \mathrm{Aut}(A_{n,\mathbf{k}})$ *be an automorphism corresponding to the polynomial* $F \in \mathbf{k}[x_1, \ldots, x_n]$ *(as in Section 4):*

$$T_F^A(\hat{x}_i) = \hat{x}_i, \ T_F^A(\hat{x}_{n+i}) = \hat{x}_{n+i} + \partial_i F(\hat{x}_1, \ldots, \hat{x}_n), \ 1 \leq i \leq n \ .$$

Then one has

$$f^{centr}(y_i) = y_i, \ f^{centr}(y_{n+i}) = y_{n+i} + \mathrm{Fr}_*(\partial_i F)(y_1, \ldots, y_n), \ 1 \leq i \leq n \ ,$$

where the polynomial $\mathrm{Fr}_*(\partial_i F) \in \mathbf{k}[y_1, \ldots, y_n]$ *is obtained from* $\partial_i F$ *by raising all coefficients to the p-th power and by replacing the variable* x_j *by* y_j*,* $1 \leq j \leq n$*.*

Proof: Let us prove the following identity for the case of one variable:

$$\left(\frac{d}{dx} + g'\right)^p = \left(\frac{d}{dx}\right)^p + (g')^p \pmod{p} \ ,$$

where R is an arbitrary ring over $\mathbb{Z}/p\mathbb{Z}$, and $g \in R[x]$ is any polynomial, $g' := dg/dx$.

A straightforward calculation over $\mathbb{Z}$ (replace the prime p by an integer and use induction) gives

$$\left(\frac{d}{dx} + g'\right)^p = \sum_{\substack{i \geq 0;\, a_1, \dots, a_p \geq 0 \\ i + \sum_j j a_j = p}} \frac{p!}{i! \prod_{j=1}^p (j!)^{a_j} \prod_{j=1}^p a_j!} \prod_{j=1}^p \left(g^{(j)}\right)^{a_j} \left(\frac{d}{dx}\right)^i .$$

Here $g^{(j)}$ denotes the j-th derivative of the polynomial g. All the coefficients above are divisible by p, except for 3 terms:

$$\left(\frac{d}{dx} + g'\right)^p = \left(\frac{d}{dx}\right)^p + (g')^p + g^{(p)} \pmod{p} .$$

The last term vanishes because $g^{(p)} = (d/dx)^p(g) = 0$ in characteristic p.

For a given $i,\ 1 \leq i \leq n$ we apply the above identity to

$$R := \mathbf{k}[\hat{x}_1, \dots, \hat{x}_{i-1}, \hat{x}_{i+1}, \dots, \hat{x}_n],\ \ g(x) := F(\hat{x}_1, \dots, \hat{x}_{i-1}, x, \hat{x}_{i+1}, \dots, \hat{x}_n)\,,$$

and get the statement of the Proposition. $\square$

It follows immediately from Propositions 2 and 3 that ϕ^{can} is well defined on symplectic linear transformations and transvections T_F^A, and hence on all tame automorphisms of $A_{n,\mathbb{C}}$. Also it is clear that the following inclusion holds

$$\operatorname{Ker} \rho_n^A \subset \operatorname{Ker} \rho_n^P$$

(with the notation introduced in Section 4). Namely, let us assume that the composition of a sequence of elementary tame automorphisms of $A_{n,\mathbb{C}}$ is the identity morphism. The corresponding transformation of centers of the Weyl algebra in large finite characteristics is the composition (twisted by Frobenius) of the *same* elementary tame automorphisms applied to P_n. Hence, the composition of elementary tame automorphisms of $P_{n,\mathbb{C}}$ is an identity.

Conversely, let us assume that a composition of elementary tame transformations of $P_{n,\mathbb{C}}$ is not an identity. Then applying Lemma 4 we obtain that the corresponding composition in $\operatorname{Aut}(A_{n,\mathbb{C}})$ is not an identity. Thus,

$$\operatorname{Ker} \rho_n^P \subset \operatorname{Ker} \rho_n^A .$$

Theorem 1 is proven. $\square$

It is an interesting challenge to find a different proof of Theorem 1, without arguments coming from finite characteristic.

8 Conjecture for the inverse map

Up to now we talked only about a homomorphism from $\mathrm{Aut}(A_{n,\mathbb{C}})$ to $\mathrm{Aut}(P_{n,\mathbb{C}})$, and never about the inverse map. Here we propose a hypothetical construction which produce an automorphism of the Weyl algebra starting from a polynomial symplectomorphism.

8.1 Brauer group and 1-forms

It is well known that for any Noetherian scheme S in characteristic $p > 0$ there exists a canonical map

$$\alpha : \Omega^1_{abs}(S)/d\mathcal{O}(S) \to \mathrm{Br}(S)$$

where $\Omega^1_{abs}(S) := \Gamma(S, \Omega_{S/\mathsf{Spec}\,\mathbb{Z}/p\mathbb{Z}})$ is the space of global absolute Kähler differentials on S. Let us assume for simplicity that S is affine. For any two functions $f, g \in \mathcal{O}(S)$ we define an associative algebra $A_{f,g}$ over S by generators and relations

$$A_{f,g} := \mathcal{O}(S)\langle \xi, \eta \rangle / (\text{ relations } [\xi, \eta] = 1,\ \xi^p = f,\ \eta^p = g)\ .$$

It is easy to see that $A_{f,g}$ is an Azumaya algebra of rank p. The correspondence α is given by

$$\alpha\left(\sum_i f_i dg_i\right) := \sum_i [A_{f_i,g_i}] = [\otimes_i A_{f_i,g_i}]\ ,$$

where $[A_{f_i,g_i}] \in \mathrm{Br}(S)$ is the class of algebra A_{f_i,g_i} in the Brauer group, which is by definition the set of equivalence classes of Azumaya algebras over S modulo Morita equivalences identical over centers $\simeq \mathcal{O}(S)$.

It follows directly from the definitions that for any commutative ring R over $\mathbb{Z}/p\mathbb{Z}$ one has an isomorphism of algebras over $C_{n,R} = \mathrm{Center}(A_{n,R}) \simeq R[y_1, \ldots, y_{2n}]$:

$$A_{n,R} \simeq \otimes_{C_{n,R}} A_{y_i, y_{n+i}}\ .$$

Hence the class $[A_{n,R}] \in \mathrm{Br}(C_{n,R})$ is given by 1-form

$$\beta_n := \sum_{i=1}^{n} y_i dy_{n+i} \pmod{dC_{n,R}} .$$

The correctness of the definition of the map α follows from the existence of certain bimodules establishing the Morita equivalences.

8.1.1 Explicit Morita equivalences

One can construct explicitly the following isomorphisms of $\mathcal{O}(S)$-algebras:

- $A_{f,0} \simeq A_{0,g} \simeq \mathrm{Mat}(p \times p, \mathcal{O}(S))$
- $A_{f,g} \simeq A_{g,-f}$ (Fourier transform)
- $A_{f_1+f_2,g} \otimes_{\mathcal{O}(S)} \mathrm{Mat}(p \times p, \mathcal{O}(S)) \simeq A_{f_1,g} \otimes_{\mathcal{O}(S)} A_{f_2,g}$
- $A_{f,g_1+g_2} \otimes_{\mathcal{O}(S)} \mathrm{Mat}(p \times p, \mathcal{O}(S)) \simeq A_{f,g_1} \otimes_{\mathcal{O}(S)} A_{f,g_2}$
- $A_{f,gh} \otimes_{\mathcal{O}(S)} A_{g,hf} \otimes_{\mathcal{O}(S)} A_{fh,g} \simeq Mat(p^3 \times p^3, \mathcal{O}(S))$
- $A_{1,f} \simeq \mathrm{Mat}(p \times p, \mathcal{O}(S))$

corresponding to basic identities in $\Omega^1_{abs}(S)/d\mathcal{O}(S)$:

- $f\, d(0) = 0\, dg = 0$
- $f\, dg = g\, d(-f) \in \Omega^1_{abs}(S)/d\mathcal{O}(S)$
- $(f_1 + f_2)\, dg = f_1 dg + f_2 dg$
- $f\, d(g_1 + g_2) = f\, dg_1 + f\, dg_2$
- $f\, d(gh) + g\, d(hf) + h\, d(fg) = 0$
- $1\, df = 0 \in \Omega^1_{abs}(S)/d\mathcal{O}(S)$

It is convenient to replace the matrix algebra $\mathrm{Mat}(p \times p, \mathcal{O}(S))$ by the algebra $A_{0,0}$. For example, the isomorphism between the algebras $A_{0,0}$ and $A_{f,0}$, $\xi^p = 0, \eta^p = 0, [\xi, \eta] = 1$ and $\xi'^p = f, \eta'^p = 0, [\xi', \eta'] = 1$ corresponding

to the pairs $(f,0)$ and $(0,0)$, is given by the formula $\xi' \to \xi - f\eta^{p-1}$, $\eta' \to \eta$. Similarly, the more complicated fifth isomorphism in the above list

$$A_{f,gh}\otimes_{\mathcal{O}(S)}A_{g,hf}\otimes_{\mathcal{O}(S)}A_{h,fg} \simeq A_{f,0}\otimes_{\mathcal{O}(S)}A_{g,0}\otimes_{\mathcal{O}(S)}A_{h,0}(\simeq \mathrm{Mat}(p^{3n}\times p^{3n},\mathcal{O}(S)))$$

(here we use isomorphisms $A_{0,0} \simeq A_{f,0}$ etc.) is given by the formula

$$\xi_1' = \xi_1,\ \eta_1' = \eta_1 - \xi_2\xi_3\,,\ \xi_2' = \xi_2,\ \eta_2' = \eta_2 - \xi_3\xi_1\,,\ \xi_3' = \xi_3,\ \eta_3' = \eta_3 - \xi_1\xi_2\,.$$

We leave to the interested reader the construction of other isomorphisms as an exercise.

8.2 Pullback of the Azumaya algebra under a symplectomorphism

Let R be a finitely generated smooth commutative algebra over $\mathbb{Z}$, and $g \in \mathrm{Aut}(P_{n,R})$ be a symplectomorphism defined over R. Our goal is to construct an automorphism $f \in \mathrm{Aut}(A_{n,R\otimes Q})$ such that $\phi_R^{can}(f) = g$.

By definition, we have

$$g(\omega) = \omega$$

where $\omega = d\beta_n = \sum_{i=1}^n dx_i \wedge dx_{i+n}$ is the standard symplectic form on $\mathbb{A}_R^{2n}$. Hence there exists a polynomial $P \in \mathbb{Q}\otimes R[x_1,\dots,x_{2n}]$ such that

$$g(\beta_n) = \beta_n + dP \in \Omega^1_{\mathbb{Q}\otimes R[x_1,\dots,x_{2n}]/\mathbb{Q}\otimes R}\,,$$

the reason is that $H^1_{de\ Rham}(\mathbb{A}^{2n}_{R\otimes\mathbb{Q}}) = 0$. We can add to R inverses of finitely many primes and assume that $P \in R[x_1,\dots,x_{2n}]$.

For any prime p let us consider the symplectomorphism $\mathrm{Fr}_*(g) \in \mathrm{Aut}(P_{n,R/pR})$ which is obtained from $g_p := g \pmod p \in \mathrm{Aut}(P_{n,R/pR})$ by raising to the p-th power all the coefficients (in other words, by applying the Frobenius endomorphism $R/pR \to R/pR$).

We claim that $\mathrm{Fr}_*(g_p)$ preserves the class of β_n in

$$\Omega^1_{abs}\left(R/pR[x_1,\dots,x_{2n}]\right)/d\left(R/pR[x_1,\dots,x_{2n}]\right)\ .$$

Obviously, we have an identity

$$\mathrm{Fr}_*(g_p)(\beta_{n,p}) = \beta_{n,p} + d\ \mathrm{Fr}_*(P_p)\ \in \Omega^1_{R/pR[x_1,\dots,x_{2n}]/(R/pR)}$$

in the space of *relative* 1-forms over R/pR, where $\beta_{n,p} := \beta_n \pmod p$, $P_p := P \pmod p$. The same identity holds in *absolute* 1-forms because all the coefficients of the transformation $\mathrm{Fr}_*(g_p)$ and of the polynomial $\mathrm{Fr}_*(P_p)$ belong to the image of the Frobenius map $\mathrm{Fr}(R/pR) = (R/pR)^p \subset R/pR$, and hence behave like constants for absolute 1-forms:

$$d(a^p b) = a^p db \in \Omega^1_{abs}\left(R/pR[x_1, \dots, x_{2n}]\right), \ \ \forall a \in R/pR, \ b \in \left(R/pR[x_1, \dots, x_{2n}]\right) \ .$$

The conclusion is that the pullback of the algebra $A_{n,R/pR}$ under the symplectomorphism $\mathrm{Fr}_*(g_p)$ has the same class in $\mathrm{Br}(\mathbb{A}^{2n}_{R/pR})$ as the algebra $A_{n,R/pR}$ itself. Therefore, there exists a Morita equivalence between these two algebras, identical on the center. In other words, we proved that there exists a Morita autoequivalence of $A_{n,R/pR}$ inducing an automorphism $\mathrm{Fr}_*(g_p)$ of the center.

Let us denote by $M_{g,p}$ *any* bimodule over $A_{n,R/pR}$ corresponding to the above Morita autoequivalence. The following result shows that this bimodule is essentially unique, its isomorphism class is uniquely determined by g.

Lemma 5 *Any bimodule over $A_{n,R/pR}$ inducing Morita autoequivalence identical on the center $C_{n,R/pR}$ is isomorphic to the diagonal bimodule. Any automorphism of the diagonal bimodule is given by multiplication by a constant in $(R/pR)^\times$.*

Proof: It is easy to see that isomorphism classes of such bimodules form a torsor over $H^1_{\acute{e}t}(\mathbb{A}^{2n}_{R/pR}, \mathbb{G}_m) \simeq 0$. Similarly, symmetries of any such modules are $H^0_{\acute{e}t}(\mathbb{A}^{2n}_{R/pR}, \mathbb{G}_m) \simeq (R/pR)^\times$. $\square$

8.3 A reformulation of the conjectures

In the notation introduced above, the bimodule $M_{g,p}$ is a finitely generated projective left $A_{n,R/pR}$-module. It corresponds to an automorphism of the algebra $A_{n,R/pR}$ iff it is a free rank one module. Moreover, the corresponding automorphism is uniquely defined because all invertible elements of $A_{n,R/pR}$ are central.

Conjecture 4 is equivalent to the following

Conjecture 5 *For any finitely generated smooth commutative algebra R over $\mathbb{Z}$ and any $g \in \mathrm{Aut}(P_{n,R})$ for all sufficiently large p, the bimodule $M_{g,p}$ is a free rank one left $A_{n,R/pR}$-module.*

There is no clear evidence for this conjecture as there are examples (see [11]) of projective finitely generated modules over the Weyl algebra A_1 in characteristic zero, which are in a sense of rank 1 and not free. In other words, an analogue of the Serre conjecture for Weyl algebras is false.

If Conjectures 4 and 5 fail to be true, it is still quite feasible that the following *weaker* version of Conjecture 1 holds:

Conjecture 6 *The group of Morita autoequivalences of the algebra $A_{n,\mathbb{C}}$ is isomorphic to the group of polynomial symplectomorphisms $Aut(P_{n,\mathbb{C}})$.*

Its equivalence to Conjecture 1 depends on the answer to the following question:

Does any Morita autoequivalence of $A_{n,\mathbb{C}}$ come from an automorphism?

8.4 Adding the Planck constant

It follows from Conjecture 1 that there should exist a mysterious nontrivial action of $\mathbb{C}^\times$ by *outer* automorphisms of the group $\mathrm{Aut}(A_{n,\mathbb{C}})$:

$$\mathbb{C}^\times \to \mathrm{Out}(\mathrm{Aut}(A_{n,\mathbb{C}}))$$

corresponding to the conjugation by the dilations

$$(x_1, \dots, x_{2n}) \to (x_1, \dots, x_n, \lambda x_{n+1}, \dots, \lambda x_{2n}),\ \lambda \in \mathbb{C}^\times$$

acting by automorphisms of $\mathrm{Aut}(P_{n,\mathbb{C}}) \subset \mathrm{Aut}(\mathbb{C}[x_1, \dots, x_n]) = \mathrm{Aut}(\mathbb{A}^{2n}_{\mathbb{C}})$. Alternatively, one can say that there is a one-parameter family of hypothetical isomorphisms between $\mathrm{Aut}(A_{n,\mathbb{C}})$ and $\mathrm{Aut}(P_{n,\mathbb{C}})$.

In general, it makes sense to introduce a new central variable $\hbar$ ("Planck constant" parameterizing the above family of isomorphisms), and define the algebra $A^{\hbar}_{n,R}$ as an associative algebra over the commutative ring $R[\hbar]$ given by generators $\hat{x}_1, \dots, \hat{x}_{2n}$ and defining relations

$$[\hat{x}_i, \hat{x}_j] = \hbar\omega_{ij}\ .$$

We propose the following

Conjecture 7 *For any finitely generated smooth commutative algebra R over $\mathbb{Z}$ and any symplectomorphism $g \in \mathrm{Aut}(P_{n,R})$ there exists a positive integer M and an automorphism $\tilde{g} \in \mathrm{Aut}(A^{\hbar}_{n,R(M^{-1})[\hbar]})$ over $R(M^{-1})[\hbar]$ such that*

- $\tilde{g} \pmod{\hbar} = g$,
- *for all sufficiently large* p *the automorphism* $\tilde{g} \pmod{p}$ *preserves the subalgebra* $R/pR[y_1^p, \ldots, y_{2n}^p]$.

This conjecture seems to be the best one, one can easily see that it implies *all* the conjectures previously made in the present paper.

9 On the extensions of the conjecture to other algebras

The Weyl algebra is isomorphic to the algebra of differential operators on the affine space. It has a natural generalization, the algebra $\mathcal{D}(X)$ of differential operators on a smooth affine algebraic variety $X/\mathbf{k}$, $\mathrm{char}(\mathbf{k}) = 0$. The corresponding Poisson counterpart is the algebra of functions on the cotangent bundle T^*X endowed with the natural Poisson bracket. One may ask what happens with our conjectures for such algebras. Unfortunately, one of our key results (Theorem 2) relies heavily on the local ad-nilpotence property of the generators of $A_{n,\mathbf{k}} = \mathcal{D}(\mathbb{A}^n_{\mathbf{k}})$ which does not hold in general. Moreover, it is easy to see that there are counter-examples to the naive extension of Conjecture 1. In particular, for $X = \mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$ with invertible coordinate x, the automorphism of $D(X)$ given by

$$x \mapsto x, \;\; \partial/\partial x \mapsto \partial/\partial x + c/x$$

does not seem to correspond to any particular symplectomorphism of T^*X, as the corresponding transformation of the center in characteristic $p > 0$ is

$$x^p \mapsto x^p, \;\; (\partial/\partial x)^p \mapsto (\partial/\partial x)^p + (c^p - c)/x^p \ .$$

The constant $(c^p - c)$ does not belong to the image of the Frobenius map in general.

One can try to generalize Conjecture 1 in a different direction. Any automorphism f of $A_{n,\mathbf{k}}$ gives a bimodule, which can be interpreted as a holonomic module $M_{(f)}$ over $A_{2n,\mathbf{k}}$. Similarly, any symplectomorphism $g \in \mathrm{Aut}(P_{n,\mathbf{k}})$ gives a Lagrangian submanifold $L_{(g)} \subset \mathbb{A}^{4n}_{\mathbf{k}}$ (the graph of g). The idea is to establish a correspondence between holonomic $\mathcal{D}$-modules and Lagrangian subvarieties (a version of the Hitchin correspondence).

In general, any holonomic module M over $D(X)$ defined over a finitely generated smooth ring $R \subset \mathbb{C}$, gives a family of coherent sheaves M_p over $\mathrm{Fr}_p^*(T^*X)$ by considering reductions modulo p as modules over the centers of algebras of differential operators. One expects (in analogy with the theory of characteristic varieties) that the support of M_p is a Lagrangian subvariety L_p of the twisted by Frobenius cotangent space, at least for large p (see the recent preprint [7] where the analog of the Hitchin correspondence was studied for a *fixed* prime p). The subvariety L_p can be singular and not necessarily conical. Moreover, its dependence on p in general seems to be chaotic.

Nevertheless, we expect that in certain circumstances there is a canonical correspondence in characteristic zero. Namely, we believe that for any closed Lagrangian subvariety L of T^*X such that L is smooth and $H^1(L(\mathbb{C}), \mathbb{Z}) = 0$, there exists a canonical holonomic $D(X)$-module M_L such that $(M_L)_p$ is supported on $\mathrm{Fr}_p(L)$ and moreover, is locally isomorphic to the sum of $p^{dim\, X}$ copies of $\mathcal{O}_{\mathrm{Fr}_p(L)}$. This would imply Conjecture 6.

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